

Note: For the benefit of the students, specially the aspiring ones, the question of JEE(advanced), 2020 are also given in this booklet. Keeping the interest of students studying in class XI, the questions based on topics from class XI have been marked with '*', which can be attempted as a test. For this test the time allocated in Physics, Chemistry & Mathematics are 20 minutes, 30 minutes and 25 minutes respectively.

FIITJEE

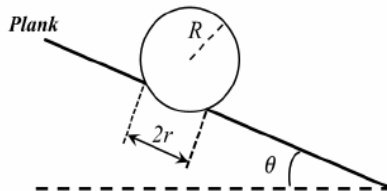
SOLUTIONS TO JEE (ADVANCED) – 2020

PART I: PHYSICS

Section - 1 (Maximum Marks: 18)

- This section contains **SIX (06)** questions.
- Each question have **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:
Full Marks : +3 If **ONLY** the correct option is chosen.
Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
Negative Marks : -1 In all other cases.

- *1. A football of radius R is kept on a hole of radius r ($r < R$) made on a plank kept horizontally. One end of the plank is now lifted so that it gets tilted making an angle θ from the horizontal as shown in the figure below. The maximum value of θ so that the football does not start rolling down the plank satisfies (figure is schematic and not drawn to scale)



(A) $\sin \theta = \frac{r}{R}$

(B) $\tan \theta = \frac{r}{R}$

(C) $\sin \theta = \frac{r}{2R}$

(D) $\cos \theta = \frac{r}{2R}$

Sol.

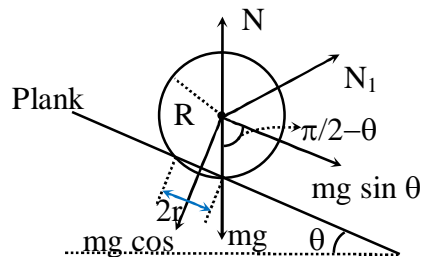
A

At the verge of toppling, $N_1 = 0$

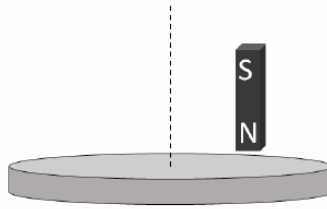
$$mg \sin \theta h = mg \cos \theta r \text{ and } \cos \theta = \frac{h}{R}$$

$$\text{So, } mg \sin \theta h = mg \frac{h}{R} r$$

$$\Rightarrow \sin \theta = r/R$$



2. A light disc made of aluminium (a nonmagnetic material) is kept horizontally and is free to rotate about its axis as shown in the figure. A strong magnet is held vertically at a point above the disc away from its axis. On revolving the magnet about the axis of the disc, the disc will (figure is schematic and not drawn to scale)

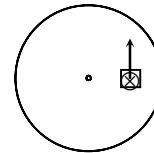


- (A) rotate in the direction opposite to the direction of magnet's motion
 (B) rotate in the same direction as the direction of magnet's motion
 (C) not rotate and its temperature will remain unchanged
 (D) not rotate but its temperature will slowly rise

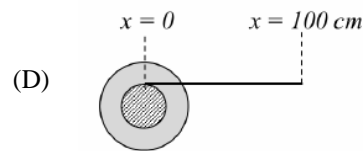
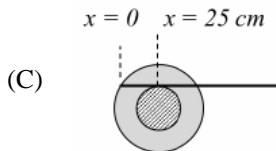
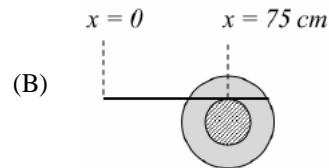
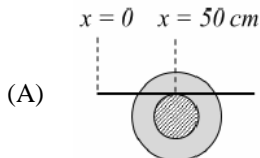
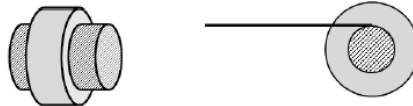
Sol.

B

⊗ is decreasing, so clockwise current in the square loop is induced. So force (along the arrow) is acting on the disc. So movement of disc is along the direction of motion of magnet.



- *3. A small roller of diameter 20 cm has an axle of diameter 10 cm (see figure below on the left). It is on a horizontal floor and a meter scale is positioned horizontally on its axle with one edge of the scale on top of the axle (see figure on the right). The scale is now pushed slowly on the axle so that it moves without slipping on the axle, and the roller starts rolling without slipping. After the roller has moved 50 cm, the position of the scale will look like (figures are schematic and not drawn to scale)



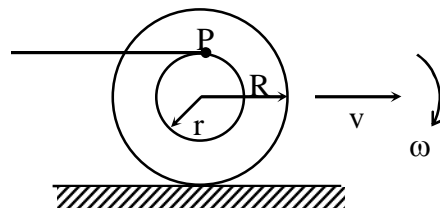
Sol.

B

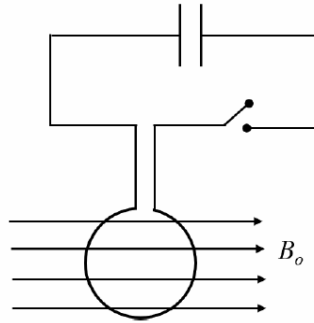
$$v - \omega R = 0$$

$$v_p = v + \omega r = v + \frac{v}{R} r = v \left(1 + \frac{r}{R} \right)$$

$$\text{So, } \Delta x_p = 50 \left(1 + \frac{5}{10} \right) = 75 \text{ cm}$$



4. A circular coil of radius R and N turns has negligible resistance. As shown in the schematic figure, its two ends are connected to two wires and it is hanging by those wires with its plane being vertical. The wires are connected to a capacitor with charge Q through a switch. The coil is in a horizontal uniform magnetic field B_0 parallel to the plane of the coil. When the switch is closed, the capacitor gets discharged through the coil in a very short time. By the time the capacitor is discharged fully, magnitude of the angular momentum gained by the coil will be (assume that the discharge time is so short that the coil has hardly rotated during this time)

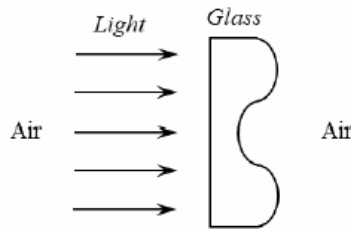


- (A) $\frac{\pi}{2} NQB_0R^2$ (B) πNQB_0R^2
 (C) $2\pi NQB_0R^2$ (D) $4\pi NQB_0R^2$

Sol.

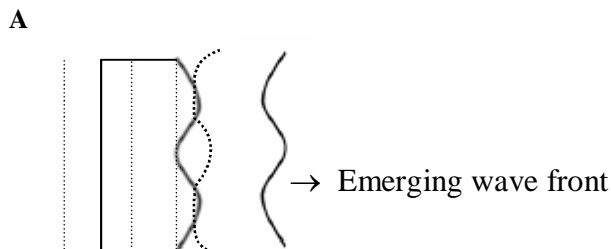
B
 $A.M. = \int \tau dt = \int NI\pi R^2 B dt = N\pi R^2 B_0 Q$

5. A parallel beam of light strikes a piece of transparent glass having cross section as shown in the figure below. Correct shape of the emergent wavefront will be (figures are schematic and not drawn to scale)



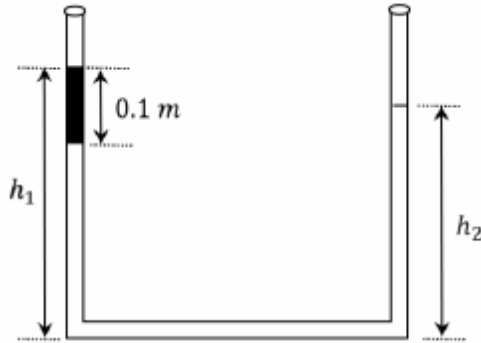
- (A) (B)
 (C) (D)

Sol.



- *6 An open-ended U-tube of uniform cross-sectional area contains water (density 10^3 kg m^{-3}). Initially the water level stands at 0.29 m from the bottom in each arm. Kerosene oil (a water-immiscible liquid) of density 800 kg m^{-3} is added to the left arm until its length is 0.1 m, as shown in the schematic figure below.

The ratio $\left(\frac{h_1}{h_2}\right)$ of the heights of the liquid in the two arms is



- (A) $\frac{15}{14}$
(C) $\frac{7}{6}$

- (B) $\frac{35}{33}$
(D) $\frac{5}{4}$

Sol.

B

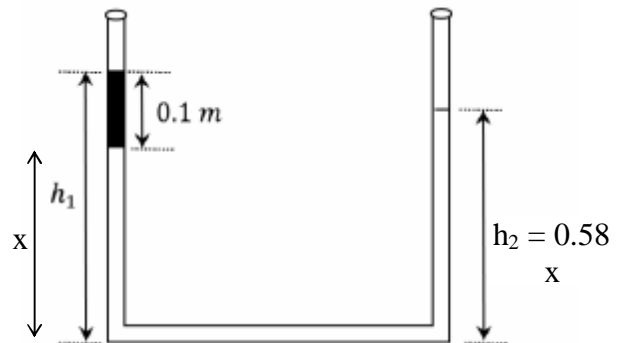
$$800 \times 0.1 + 1000 \times x = 1000 \times (0.58 - x)$$

$$2000x = 500$$

$$x = 0.25 \text{ m}$$

$$\text{So, } h_1 = 0.35 \text{ m, } h_2 = 0.33 \text{ m}$$

$$\frac{h_1}{h_2} = \frac{35}{33}$$



Section 2 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- Each question have **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) the correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:
 Full Marks : +4 If only (all) the correct option(s) is(are) chosen.
 Partial Marks : +3 all the four the correct option but **ONLY** three options are chosen.
 Partial Marks : +2 If three or more options are correct but **ONLY** two are chosen both of which are correct.
 Partial Marks : +1 If two or more options are correct but **ONLY** one option is chosen and it is a correct option.
 Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
 Negative Marks : -2 In all other cases.

7. A particle of mass m moves in circular orbits with potential energy $V(r) = Fr$, where F is a positive constant and r is its distance from the origin. Its energies are calculated using the Bohr model. If the radius of the particle's orbit is denoted by R and its speed and energy are denoted by v and E , respectively, then for the n^{th} orbit (here h is the Planck's constant)

(A) $R \propto n^{1/3}$ and $v \propto n^{2/3}$

(B) $R \propto n^{2/3}$ and $v \propto n^{1/3}$

(C) $E = \frac{3}{2} \left(\frac{n^2 h^2 F^2}{4\pi^2 m} \right)^{1/3}$

(D) $E = 2 \left(\frac{n^2 h^2 F^2}{4\pi^2 m} \right)^{1/3}$

Sol. B, C

$$F = -\frac{dV}{dr}$$

$$\frac{mv^2}{R} = F$$

$$mvr = \frac{nh}{2\pi}$$

Solving above equations

$$R^3 = \frac{n^2 h^2}{4\pi^2 m}$$

$$v \propto n^{1/3}$$

8. The filament of a light bulb has surface area 64 mm^2 . The filament can be considered as a black body at temperature 2500 K emitting radiation like a point source when viewed from far. At night the light bulb is observed from a distance of 100 m . Assume the pupil of the eyes of the observer to be circular with radius 3 mm . Then

(Take Stefan-Boltzmann constant $= 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$, Wien's displacement constant $= 2.90 \times 10^{-3} \text{ m-K}$, Planck's constant $= 6.63 \times 10^{-34} \text{ Js}$, speed of light in vacuum $= 3.00 \times 10^8 \text{ ms}^{-1}$)(A) power radiated by the filament is in the range 642 W to 645 W (B) radiated power entering into one eye of the observer is in the range $3.15 \times 10^{-8} \text{ W}$ to $3.25 \times 10^{-8} \text{ W}$ (C) the wavelength corresponding to the maximum intensity of light is 1160 nm (D) taking the average wavelength of emitted radiation to be 1740 nm , the total number of photons entering per second into one eye of the observer is in the range 2.75×10^{11} to 2.85×10^{11}

Sol. B, C, D

$$P = A\sigma T^4$$

$$= 64 \times 10^{-6} \times 5.67 \times 10^{-8} \times (2500)^4 \approx 140 \text{ W}$$

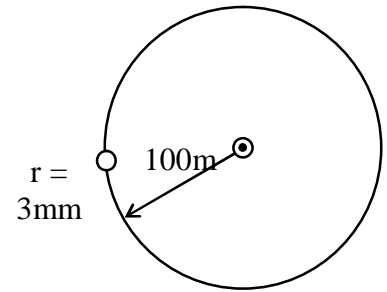
Power entering in eye

$$= \frac{140}{4\pi(100)^2} \pi(3 \times 10^{-3})^2 = \frac{140 \times 9 \times 10^{-10}}{4} = 3.15 \times 10^{-8} \text{ W}$$

$$\lambda T = b \Rightarrow \lambda = \frac{2.93 \times 10^{-6}}{2500} \Rightarrow \lambda = 1160$$

Photons

$$n \frac{12420}{1740} = \frac{3.15 \times 10^{-8}}{1.6 \times 10^{-19}} = 2.75 \times 10^{11}$$



9. Sometimes it is convenient to construct a system of units so that all quantities can be expressed in terms of only one physical quantity. In one such system, dimensions of different quantities are given in terms of a quantity X as follows: [position] = [X^α]; [Speed] = [X^β]; [acceleration] = [X^p]; [Linear momentum] = [X^q]; [force] = [X^r]. Then

(A) $\alpha + p = 2\beta$

(B) $p + q - r = \beta$

(C) $p - q + r = \alpha$

(D) $p + q + r = \beta$

Sol. A, B

$$r = x^\alpha$$

$$v = x^\beta$$

$$a = x^p$$

$$P = x^q$$

$$F = x^r$$

(A) $ra \rightarrow [x^{\alpha+p}]$

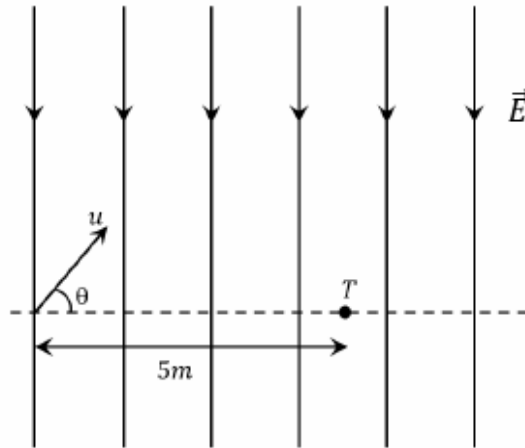
$$\left[\frac{\text{Energy}}{\text{mass}} \right] = [2\beta]$$

(B) $\left[\frac{aP}{F} \right] = [at] = [v] = \beta$

$$P + q - r = \beta$$

10. A uniform electric field, $\vec{E} = -400\sqrt{3}\hat{y} \text{ NC}^{-1}$ is applied in a region. A charged particle of mass m carrying positive charge q is projected in this region with an initial speed of $2\sqrt{10} \times 10^6 \text{ ms}^{-1}$. This particle is aimed to hit a target T, which is 5 m away from its entry point into the field as shown schematically in the figure.

Take $\frac{q}{m} = 10^{10} \text{ Ckg}^{-1}$. Then



- (A) the particle will hit T if projected at an angle 45° from the horizontal
 (B) the particle will hit T if projected either at an angle 30° or 60° from the horizontal
 (C) time taken by the particle to hit T could be $\sqrt{\frac{5}{6}} \mu\text{s}$ as well as $\sqrt{\frac{5}{2}} \mu\text{s}$
 (D) time taken by the particle to hit T is $\sqrt{\frac{5}{3}} \mu\text{s}$

Sol. B, C

$$\text{Range} = R = \frac{2u \sin \theta}{g_{\text{eff}}} u \cos \theta = 5\text{m}$$

$$\sin 2\theta = \left(\frac{qER}{mu^2} \right) = \frac{\sqrt{3}}{2}$$

$$\theta = 30^\circ$$

So for same range angle of projection is either 30° or 60°

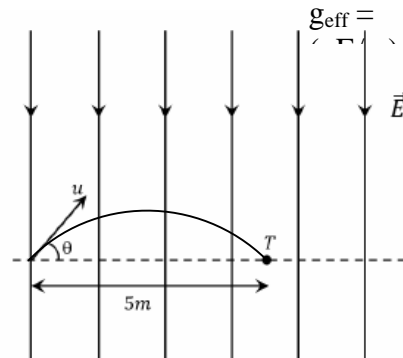
So Option (B) is correct

$$T = \frac{2u \sin \theta}{g_{\text{eff}}} = \left(\sqrt{\frac{10}{3}} \times 10^{-6} \right) \sin \theta$$

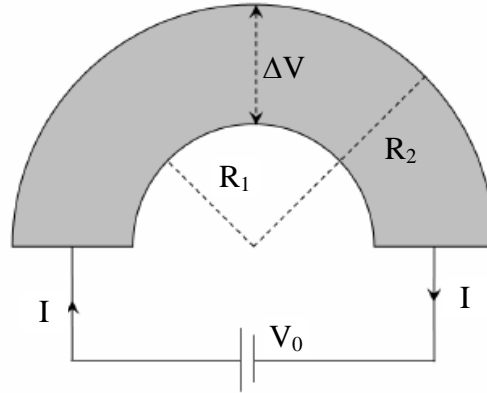
$$T_1(\theta = 30^\circ) = \sqrt{\frac{5}{6}} \mu\text{s}$$

$$T_2(\theta = 60^\circ) = \sqrt{\frac{5}{2}} \mu\text{s}$$

So, Option (C) is correct



- 11 Shown in the figure is a semicircular metallic strip that has thickness t and resistivity ρ . Its inner radius is R_1 and outer radius is R_2 . If a voltage V_0 is applied between its two ends, a current I flows in it. In addition, it is observed that a transverse voltage ΔV develops between its inner and outer surfaces due to purely kinetic effects of moving electrons (ignore any role of the magnetic field due to the current). Then (figure is schematic and not drawn to scale)



- (A) $I = \frac{V_0 t}{\pi \rho} \ln\left(\frac{R_2}{R_1}\right)$
- (B) the outer surface is at a higher voltage than the inner surface
- (C) the outer surface is at a lower voltage than the inner surface
- (D) $\Delta V \propto I^2$

Sol. A, C, D

Area of the strip = $(t \cdot dr)$

$$dR = \rho \left(\frac{\pi r}{t dr} \right)$$

$$\sum \frac{1}{dR} = \int_{R_1}^{R_2} \frac{t dr}{\pi \rho r}$$

$$\frac{1}{R_{eq}} = \left(\frac{t}{\pi \rho} \right) \ln\left(\frac{R_2}{R_1}\right)$$

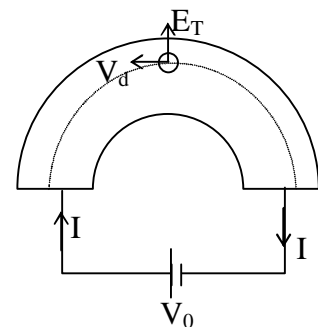
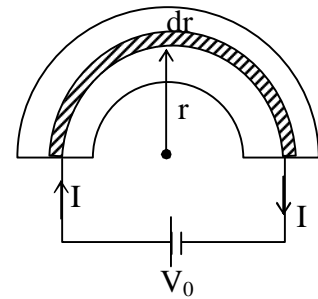
$$R_{eq} = \frac{\pi \rho}{t \times \ln\left(\frac{R_2}{R_1}\right)}$$

So, $I = \frac{V_0}{R_{eq}} = \left(\frac{V_0 t}{\pi \rho} \right) \ln\left(\frac{R_2}{R_1}\right)$ So, option (A) is correct.

Electrons are revolving in a circular path so there is a centripetal force on the electron, which provide a transverse electric field.

The direction of transverse electric field is radially out side hence inner surface is at higher potential.

So, option (C) is correct.



$$\text{Current density } J(r) = \frac{dl}{(t dr)}$$

$$J(r) = \frac{V_0}{(dr)(t dr)}$$

$$J(r) = \frac{V_0}{\left(\frac{\rho \pi r}{t dr}\right)(t dr)}$$

$$J(r) = \frac{V_0}{(\rho \pi r)}$$

$$I = neAV_d$$

$$J(r) = neV_d$$

$$\Rightarrow V_d = \left(\frac{J(r)}{ne}\right) = \left(\frac{V_0}{\rho \pi r ne}\right) \quad \dots(i)$$

Centripetal force

$$eE_T = \frac{m_e V_d^2}{r} \quad \dots(ii)$$

$$\text{From (i) and (ii) } E_T \propto V_0^2 \quad \dots(iii)$$

$$\Delta V = \int E_T dr$$

$$\Rightarrow \Delta V \propto E_T \quad \dots(iv)$$

From (iii) and (iv)

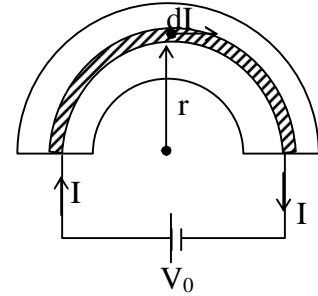
$$\Delta V \propto V_0^2$$

$$I \propto V_0$$

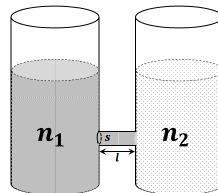
Hence

$$\Delta V \propto I^2$$

Hence Option (D) is correct.



- *12 As shown schematically in the figure, two vessels contain water solutions (at temperature T) of potassium permanganate (KMnO_4) of different concentrations n_1 and n_2 ($n_1 > n_2$) molecules per unit volume with $\Delta n = (n_1 - n_2) \ll n_1$. When they are connected by a tube of small length ℓ and cross-sectional area S , KMnO_4 starts to diffuse from the left to the right vessel through the tube. Consider the collection of molecules to behave as dilute ideal gases and the difference in their partial pressure in the two vessels causing the diffusion. The speed v of the molecules is limited by the viscous force $-\beta v$ on each molecule, where β is a constant. Neglecting all terms of the order $(\Delta n)^2$, which of the following is/are correct? (k_B is the Boltzmann constant)



- (A) the force causing the molecules to move across the tube is $\Delta n k_B T S$
 (B) force balance implies $n_1 \beta v l = \Delta n k_B T$
 (C) total number of molecules going across the tube per sec is $\left(\frac{\Delta n}{l}\right) \left(\frac{k_B T}{\beta}\right) S$
 (D) rate of molecules getting transferred through the tube does not change with time

Sol. A, B, C

$$PV = NKT$$

$$P_1 = n_1 K_B T$$

$$P_2 = n_2 K_B T$$

$n_1, n_2 \rightarrow$ no. of molecules per unit volume

Force acting on the molecules

$$F = \Delta PS$$

$$= (P_1 - P_2)S$$

$$F = (n_1 - n_2)K_B TS = \Delta n k_B TS \dots (i)$$

So Option (A) is correct

Total no. of molecules in the tube = $(n_1 S \ell)$

Force acting on each molecules

$$F = -\beta v$$

So total force acting on $(n_1 S \ell)$ molecules = -

$$(\beta v)(n_1 S \ell) \dots (ii)$$

From (i) and (ii)

$$\Delta n K_B TS = (\beta v)(n_1 S \ell)$$

$$\text{So, } \Delta n K_B T = (\beta v n_1 \ell) \dots (iii)$$

So, Option (B) is correct.

No. of molecules going across the tube per second

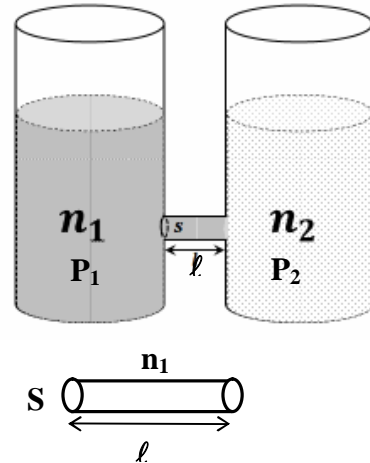
$$= (sv n_1)$$

$$= (s n_1) \left(\frac{\Delta n K_B T}{\beta n_1 \ell} \right) = \left(\frac{S \Delta n K_B T}{\beta \ell} \right)$$

So, Option (C) is correct.

With respect to time Δn changes hence rate of molecules getting transferred through the tube changes with time.

So, Option (D) is incorrect.



Section 3 (Maximum Marks: 24)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**
- For each question, enter the correct numerical value of the answer using the mouse and on-screen virtual numerical keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, truncate/round-off the value to TWO decimal places.
- Answer to each question will be evaluated according to the following marking scheme:
Full Marks : +4 If ONLY the correct numerical value is entered;
Zero Marks : 0 If all the cases.

- *13. Put a uniform meter scale horizontally on your extended index fingers with the left one at 0.00 cm and the right one at 90.00 cm. When you attempt to move both the fingers slowly towards the center, initially only the left finger slips with respect to the scale and the right finger does not. After some distance, the left finger stops and the right one starts slipping. Then the right finger stops at a distance x_R from the center (50.00 cm) of the scale and the left one starts slipping again. This happens because of the difference in the frictional forces on the two fingers. If the coefficients of static and dynamic friction between the fingers and the scale are 0.40 and 0.32, respectively, the value of x_R (in cm) is _____.

Sol. 25.6 (25.60 to 25.60)

Initially

$$40N_1 = 50N_2$$

For first move

$$\mu_k N_3 = \mu_s N_4 \quad \dots(i)$$

$$xN_3 = 40 N_4 \quad \dots(ii)$$

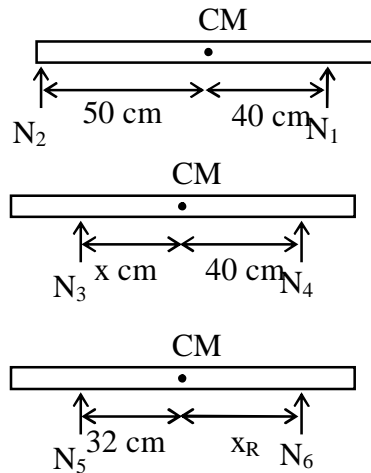
$$x = 32 \text{ cm}$$

For second move

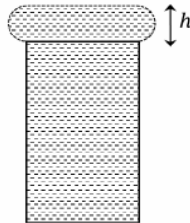
$$\mu_s N_5 = \mu_k N_6 \quad \dots(i)$$

$$32(N_5) = x_R N_6 \quad \dots(ii)$$

$$x_R = 25.6 \text{ cm}$$



- *14. When water is filled carefully in a glass, one can fill it to a height h above the rim of the glass due to the surface tension of water. To calculate h just before water starts flowing, model the shape of the water above the rim as a disc of thickness h having semicircular edges, as shown schematically in the figure. When the pressure of water at the bottom of this disc exceeds what can be withstood due to the surface tension, the water surface breaks near the rim and water starts flowing from there. If the density of water, its surface tension and the acceleration due to gravity are 10^3 kg m^{-3} , 0.07 Nm^{-1} and 10 ms^{-2} , respectively, the value of h (in mm) is _____.



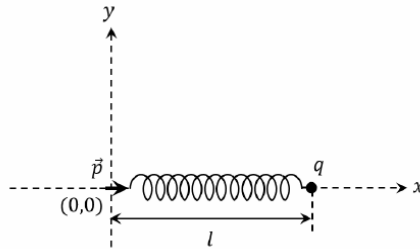
Sol. 3.75 (3.65 to 3.85)

$$S(2R) = \left(\frac{\rho g h}{2} \right) h(2R)$$

$$h = 3.75 \times 10^{-3} \text{ mm}$$

15. One end of a spring of negligible unstretched length and spring constant k is fixed at the origin $(0,0)$. A point particle of mass m carrying a positive charge q is attached at its other end. The entire system is kept on a smooth horizontal surface. When a point dipole \vec{p} pointing towards the charge q is fixed at the origin, the spring gets stretched to a length l and attains a new equilibrium position (see figure below). If the point mass is now displaced slightly by $\Delta l \ll l$ from its equilibrium position and released, it is found to oscillate

at frequency $\frac{1}{\delta} \sqrt{\frac{k}{m}}$. The value of δ is _____.



Sol. 3.14 (0.50 to 0.50) or (3.13 to 3.15)

Potential energy of a system after a small displacement.

$$U = \frac{1}{2} kx^2 + \frac{k'pq}{(\ell + x)^2}$$

$$\frac{dU}{dx} = kx - \frac{2k'pq}{(\ell + x)^3}$$

$$\frac{d^2U}{dx^2} = k + \frac{6k'pq}{(\ell + x)^4} = k + 3k = 4k$$

$$f = \frac{1}{2\pi} \sqrt{\frac{4k}{m}}$$

$$f = \frac{1}{\pi} \sqrt{\frac{k}{m}}$$

So, $\delta = 3.14$

and for angular frequency

$$\omega = \frac{1}{\delta} \sqrt{\frac{k}{m}}$$

$\delta = 0.5$

- *16. Consider one mole of helium gas enclosed in a container at initial pressure P_1 and volume V_1 . It expands isothermally to volume $4V_1$. After this, the gas expands adiabatically and its volume becomes $32V_1$. The work done by the gas during isothermal and adiabatic expansion processes are W_{iso} and W_{adia} , respectively.

If the ratio $\frac{W_{iso}}{W_{adia}} = f \ln 2$, then f is _____.

Sol. 1.77 (1.77 to 1.79)

A → B

$$P_1 V_1 = P_B \times 4V_1$$

$$\Rightarrow P_B = \frac{P_1}{4}$$

$$W_{\text{iso}} = 1 \times R \times T \ln \frac{4V_1}{V_1}$$

$$P_1 V_1 \ln 4 = 2P_1 V_1 \ln 2$$

B → C

$$P_B V_B^\gamma = P_C V_C^\gamma$$

$$\frac{P_1}{4} (4V_1)^\gamma = P_C (32V_1)^\gamma$$

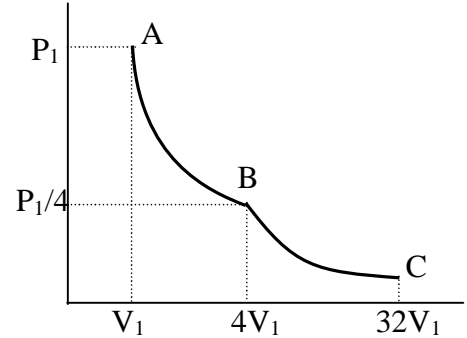
$$\Rightarrow P_C = \frac{1}{4} \left(\frac{1}{8} \right)^\gamma P_1 = \frac{1}{4} \left(\frac{1}{2^3} \right)^{5/3} P_1 = \frac{P_1}{128}$$

$$W_{\text{adio}} = \frac{\frac{P_1}{128} \times 32V_1 - \frac{P_1}{4} \times 4V_1}{1 - \frac{5}{3}}$$

$$= \frac{\frac{P_1 V_1}{4} - P_1 V_1}{-\frac{2}{3}} = \frac{-\frac{3}{4} P_1 V_1}{-\frac{2}{3}} = \frac{9}{8} P_1 V_1$$

$$\frac{W_{\text{iso}}}{W_{\text{adio}}} = \frac{2P_1 V_1 \ln 2}{\frac{9}{8} P_1 V_1} = \frac{16}{9} \ln 2$$

Comparing with given expression of = 16/9 = 1.78



- *17. A stationary tuning fork is in resonance with an air column in a pipe. If the tuning fork is moved with a speed of 2 ms^{-1} in front of the open end of the pipe and parallel to it, the length of the pipe should be changed for the resonance to occur with the moving tuning fork. If the speed of sound in air is 320 ms^{-1} , the smallest value of the percentage change required in the length of the pipe is _____.

Sol. 0.62 (0.62 to 0.64)

$$f = f_0 \frac{320}{320 - 2} = \frac{320}{318} f_0$$

$$\Rightarrow \Delta f = \frac{2}{318} f_0 \quad \dots(1)$$

$$\therefore f_0 = \frac{(2n+1)v}{4L}$$

$$\frac{2}{318} = -\frac{\Delta L}{L}$$

$$100 \times \frac{\Delta L}{L} = \frac{200}{318} = 0.62983 = 0.63$$

18. A circular disc of radius R carries surface charge density $\sigma_0(r) = \sigma_0 \left(1 - \frac{r}{R}\right)$, where σ_0 is a constant and r is the distance from the center of the disc. Electric flux through a large spherical surface that encloses the charged disc completely is ϕ_0 . Electric flux through another spherical surface of radius $\frac{R}{4}$ and concentric with the disc is ϕ . Then the ratio $\frac{\phi_0}{\phi}$ is _____

Sol. **6.40 (6.40 to 6.40)**

Total charge on disc

$$q_0 = \int_0^R \sigma_0 \left(1 - \frac{r}{R}\right) 2\pi r dr = \sigma_0 \cdot 2\pi \left[\int_0^R r dr - \frac{1}{R} \int_0^R r^2 dr \right]$$

$$= 2\pi\sigma_0 \left[\frac{R^2}{2} - \frac{R^2}{3} \right]$$

$$q_0 = 2\pi\sigma_0 \left[\frac{R^2}{6} \right] \quad \dots(1)$$

Charge enclosed by second sphere

$$q = \sigma_0 \cdot 2\pi \left[\int_0^{R/4} r dr - \frac{1}{R} \int_0^{R/4} r^2 dr \right]$$

$$= 2\pi\sigma_0 \left[\frac{R^2}{32} - \frac{R^2}{192} \right]$$

$$q = 2\pi\sigma_0 \left[\frac{5R^2}{192} \right]$$

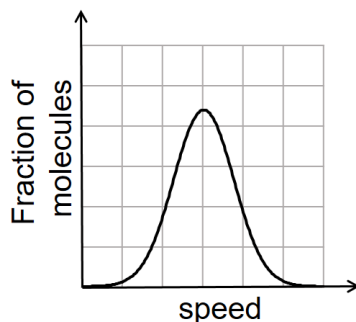
$$\Rightarrow \frac{\phi_0}{\phi} = \frac{q_0}{q} = \frac{R^2/6}{5R^2/192} = \frac{192}{5 \times 6} = \frac{32}{5} = 6.40$$

PART II: CHEMISTRY

SECTION-1 (Maximum Marks: 18)

- This section contains SIX (06) questions
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following making scheme:
 Full Marks : +3 If **ONLY** the correct option is chosen;
 Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
 Negative Marks : -1 In all other cases

- *1. If the distribution of molecular speeds of a gas is as per the figure shown below, then the ratio of the most probable, the average, and the root mean square speeds, respectively, is



- (A) 1:1:1
 (B) 1 : 1 : 1.224
 (C) 1 : 1.128 : 1.224
 (D) 1 : 1.128 : 1

Sol.

B

Since the graph is symmetrical about the peak. So, the average speed and most probable speed will coincide

but rms speed will remain $\sqrt{\frac{3RT}{M}}$

$$\text{So, } u_{mp} = \sqrt{\frac{2RT}{M}} = u_{av}$$

$$u_{rms} = \sqrt{\frac{3RT}{M}}$$

$$u_{mp} : u_{av} : u_{rms} = \sqrt{2} : \sqrt{2} : \sqrt{3} = 1 : 1 : \sqrt{\frac{3}{2}}$$

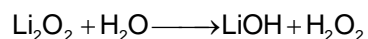
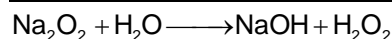
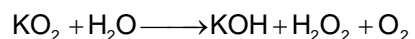
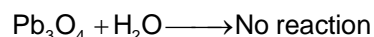
$$= 1 : 1 : 1.224$$

- *2. Which of the following liberates O_2 upon hydrolysis?

- (A) Pb_3O_4 (B) KO_2
 (C) Na_2O_2 (D) Li_2O_2

Sol.

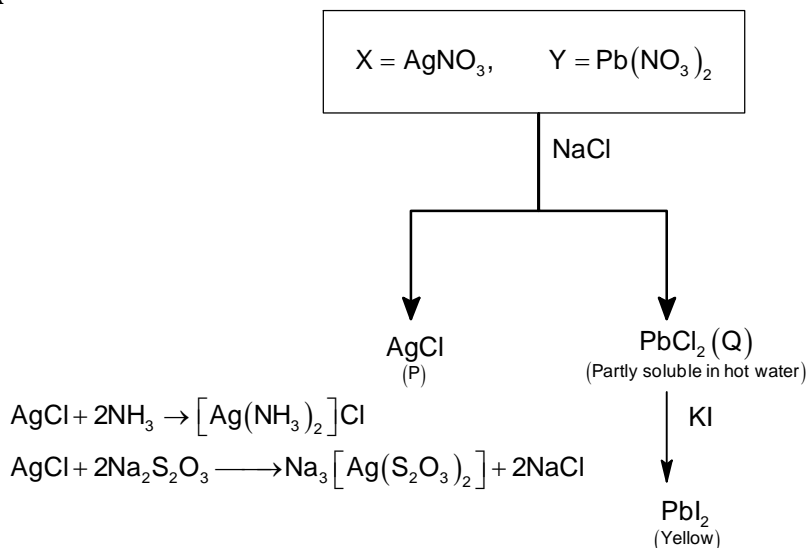
B



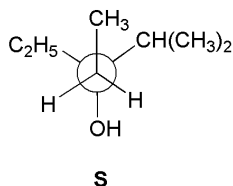
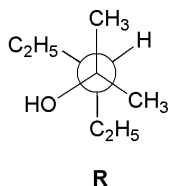
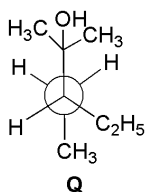
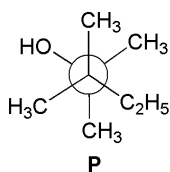
3. A colorless aqueous solution contains nitrates of two metals, **X** and **Y**. When it was added to an aqueous solution of NaCl, a white precipitate was formed. This precipitate was found to be partly soluble in hot water to give a residue **P** and a solution **Q**. The residue **P** was soluble in aqueous NH_3 and also in excess sodium thiosulfate. The hot solution **Q** gave a yellow precipitate with KI. The metals **X** and **Y**, respectively are

- (A) Ag and Pb (B) Ag and Cd
(C) Cd and Pb (D) Cd and Zn

Sol. A



- *4. Newman projections P, Q, R and S are shown below:

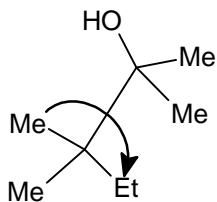


Which one of the following options represents identical molecules?

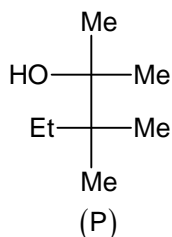
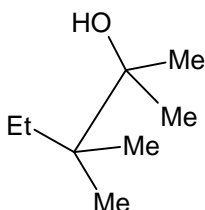
- (A) P and Q (B) Q and S
(C) Q and R (D) R and S

Sol. C

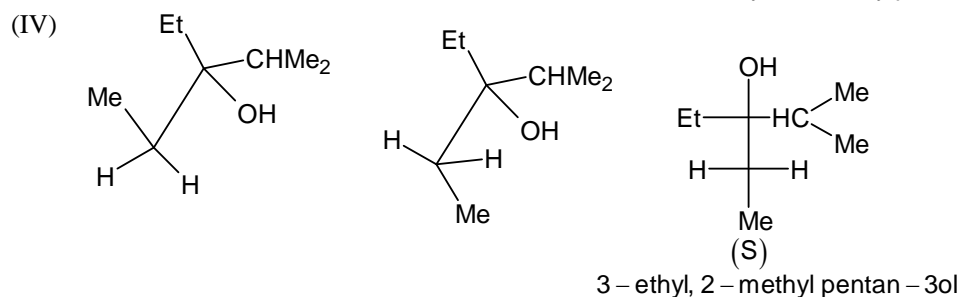
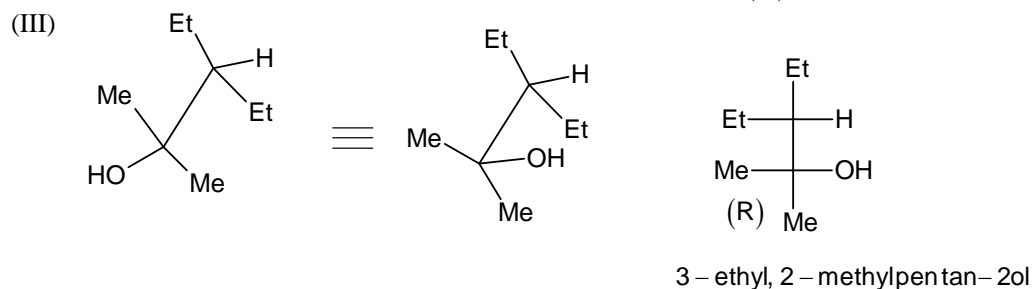
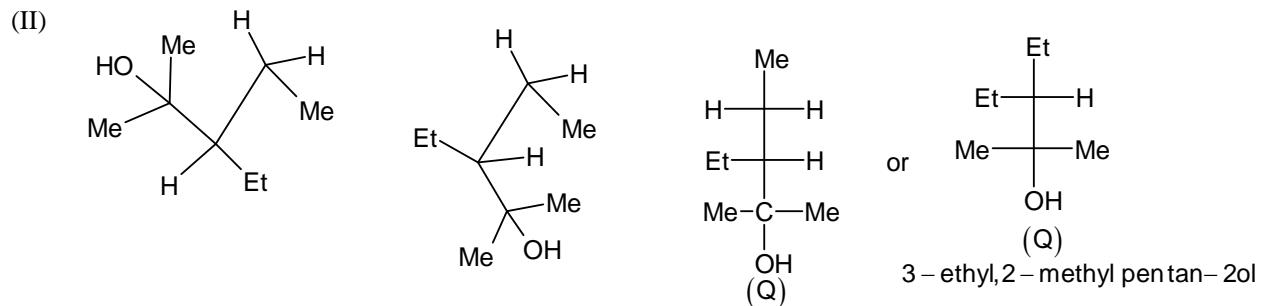
(I)



≡

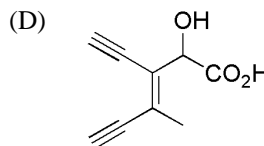
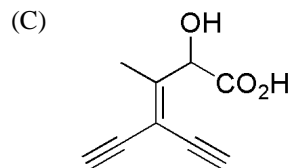
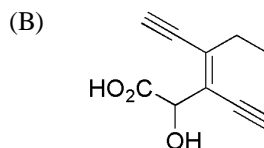
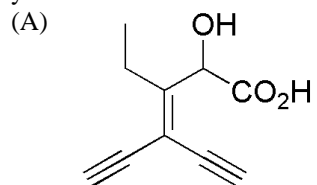


2,3,3 – trimethyl pentan – 2 ol

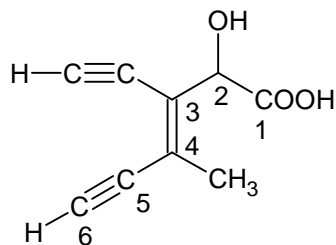


Only (Q & R) are same.

*5. Which one of the following structures has the IUPAC name 3-ethynyl-2-hydroxy-4-methylhex-3-en-5-ynoic acid?

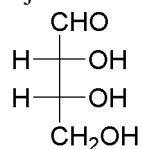


Sol.



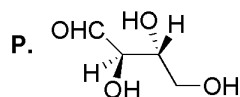
3-ethynyl, 2-hydroxy, 4-methyl hex-3-en-5-ynoic acid

*6. The Fischer projection of D-erythrose is shown below:



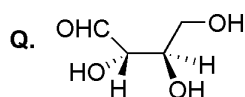
D-Erythrose

D—Erythrose and its isomers are listed as P, Q, R and S in **Column-I**. Choose the correct relationship of P, Q, R and S with D-erythrose from **Column-II**.

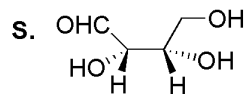
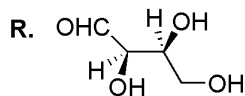
Column-I**Column-II**

1. Diastereomer

2. Identical



3. Enantiomer



(A) P → 2, Q → 3, R → 2, S → 2

(B) P → 3, Q → 1, R → 1, S → 2

(C) P → 2, Q → 1, R → 1, S → 3

(D) P → 2, Q → 3, R → 3, S → 1

Sol.**C**

Given structure : (2R, 3R)

P ⇒ (2R, 3R), so, identical. Hence P → 2

Q ⇒ (2S, 3R), so, Diastereomers. Hence Q → 1

R ⇒ (2R, 3S), so, Diastereomers. Hence R → 1

S ⇒ (2S, 3S), so, Enantiomers. Hence S → 3

SECTION-2 (Maximum Marks: 24)

- This section contains **SIX (06)** questions
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is/are correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following making scheme:

Full Marks	: +4	If Only (all) the correct option(s) is/are chosen;
Partial Marks	: +3	If all the four option are correct but ONLY three options are chosen;
Partial Marks	: +2	If three or more option are correct but ONLY two options are chosen, both of which are correct
Partial Marks	: +1	If two or more option are correct but ONLY one option are chosen, and it is a correct option
Zero Marks	: 0	If none of the options is chosen (i.e. the question is unanswered);
Negative Marks	: -2	In all other cases

*7. In thermodynamics, the P – V work done is given by

$$w = -\int dVP_{\text{ext}}$$

For a system undergoing a particular process, the work done is

$$w = -\int dV \left(\frac{RT}{V-b} - \frac{a}{V^2} \right)$$

This equation is applicable to a

- (A) system that satisfies the van der Waal's equation of state
- (B) process that is reversible and isothermal
- (C) process that is reversible and adiabatic
- (D) process that is irreversible and at constant pressure

Sol. A, B, C

$$W_{\text{rev}} = -\int PdV$$

For one mole of a real gas,

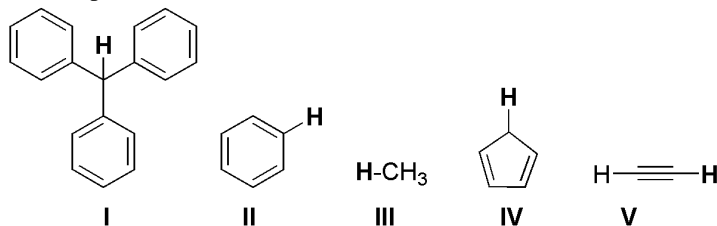
$$\left(P + \frac{a}{V^2} \right) (V - b) = RT$$

$$P = \left(\frac{RT}{(V-b)} - \frac{a}{V^2} \right)$$

$$\text{So, } W_{\text{rev}} = -\int \left(\frac{RT}{(V-b)} - \frac{a}{V^2} \right) dV \dots (1)$$

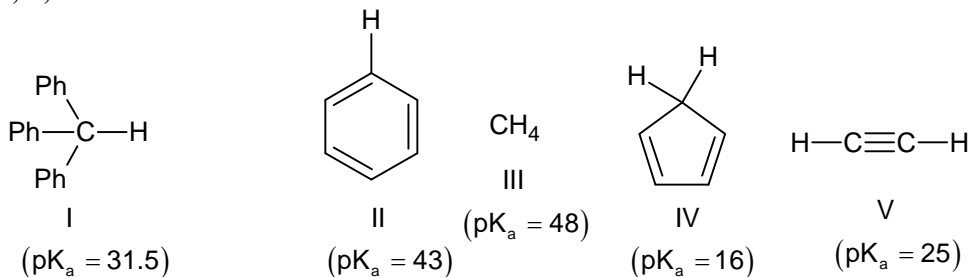
So, above Equation is valid for all reversible processes and for real gases.

*8. With respect to the compounds **I – V**, choose the correct statement(s).



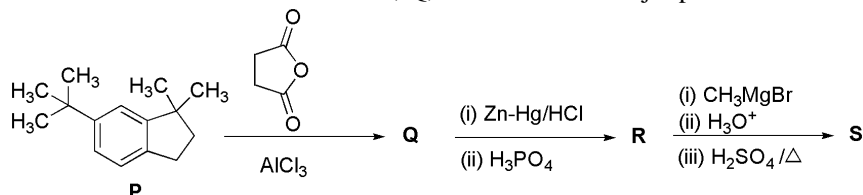
- (A) The acidity of compound I is due to delocalization in the conjugate base
- (B) The conjugate base of compound IV is aromatic
- (C) Compound II becomes more acidic, when it has a $-\text{NO}_2$ substituent
- (D) The acidity of compounds follows the order $\text{I} > \text{IV} > \text{V} > \text{II} > \text{III}$

Sol. A, B, C



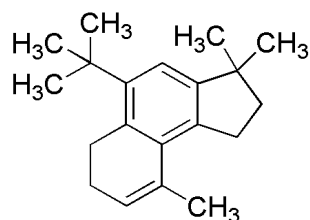
Correct order and acidity = IV > V > I > II > III (Decreasing acidity)

*9. In the reaction scheme shown below, Q, R and S are the major products.

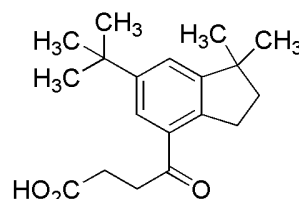


The correct structure of

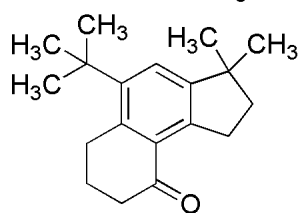
(A) S is



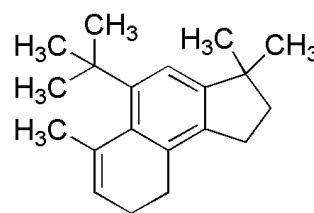
(B) Q is



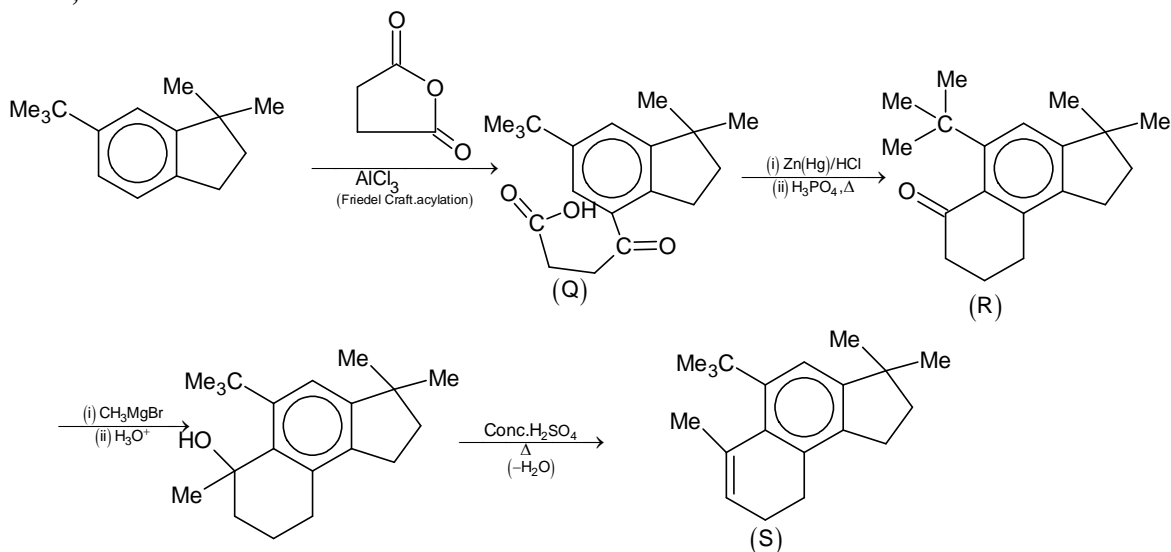
(C) R is



(D) S is



Sol. B, D

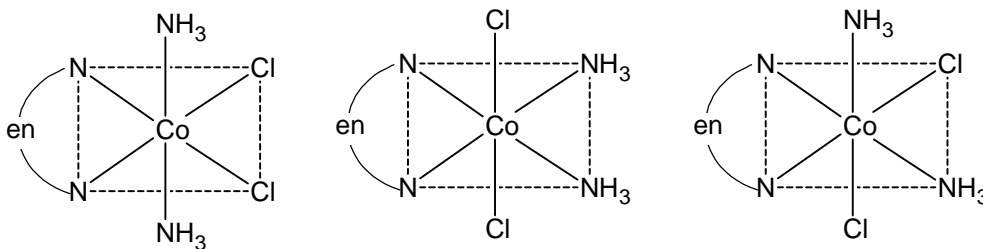


10. Choose the correct statement(s) among the following:
- (A) $[\text{FeCl}_4]^-$ has tetrahedral geometry
- (B) $[\text{Co}(\text{en})(\text{NH}_3)_2\text{Cl}_2]^+$ has 2 geometrical isomers
- (C) $[\text{FeCl}_4]^-$ has higher spin – only magnetic moment than $[\text{Co}(\text{en})(\text{NH}_3)_2\text{Cl}_2]^+$
- (D) The cobalt ion in $[\text{Co}(\text{en})(\text{NH}_3)_2\text{Cl}_2]^+$ has sp^3d^2 hybridization

Sol. A, C

(A) $[\text{FeCl}_4]^-$: Fe^{3+} , weak ligand, so tetrahedral.

(B) $[\text{Co}(\text{en})(\text{NH}_3)_2\text{Cl}_2]^+$: Three geometrical isomers.



(C) $[\text{FeCl}_4]^-$: $N = 5$. $\mu = \sqrt{5(5+2)} = \sqrt{35}$ BM

$[\text{Co}(\text{en})(\text{NH}_3)_2\text{Cl}_2]^+$: Co^{3+} (d^6), diamagnetic

(D) $[\text{Co}(\text{en})(\text{NH}_3)_2\text{Cl}_2]^+$: d^2sp^3 . $N = 0, \mu = 0$ & $[\text{FeCl}_4]^- \Rightarrow$ Tetrahedral (sp^3)

11. With respect to hypochlorite, chlorate and perchlorate ions, choose correct statement(s).
- (A) The hypochlorite ion is strongest conjugate base
- (B) The molecular shape of only chlorate ion is influenced by the lone pair of electrons of Cl
- (C) The hypochlorite and chlorate ions disproportionate to give rise to identical set of ions
- (D) The hypochlorite ion oxidizes the sulfite ion

Sol. A, B, D

(A) $\text{HClO} < \text{HClO}_3 < \text{HClO}_4$ (increasing acidic character)

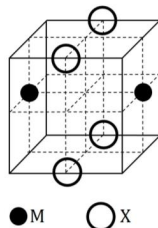
So, ClO^- is the strongest conjugate base.

(B) Only $\text{Cl}-\text{O}^-$ and ClO_3^- has lone pair of electrons on Cl-atom. But $\text{Cl}-\text{O}^-$ is always linear ion as it is diatomic, so, no effect of lone pair on shape but in ClO_3^- due the lone pair on Cl shape becomes pyramidal.

(C) $\left. \begin{array}{l} \text{ClO}^- \longrightarrow \text{Cl}^- + \text{ClO}_3^- \\ \text{ClO}_3^- \longrightarrow \text{Cl}^- + \text{ClO}_4^- \end{array} \right\}$ (No same set of ions)

(D) $\text{ClO}^- + \text{SO}_3^{2-} \longrightarrow \text{Cl}^- + \text{SO}_4^{2-}$

12. The cubic unit cell structure of a compound containing cation M and anion X is shown below. When compared to the anion, the cation has smaller ionic radius. Choose the correct statement(s).



- (A) The empirical formula of the compound is MX
 (B) The cation M and anion X have different coordination geometries
 (C) The ratio of M – X bond length to the cubic unit cell edge length is 0.866
 (D) The ratio of the ionic radii of cation M to anion X is 0.414

Sol. A, C

It can be assumed to be CsCl type solid (bcc)

$$\text{So, } r_+ + r_- = \frac{\sqrt{3}a}{2} = 0.866$$

Formula of solid = MX

$$0.732 \leq \frac{r_+}{r_-} \leq 0.999$$

SECTION-3 (Maximum Marks: 24)

- This section contains SIX (06) questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, truncate round-off the value to TWO decimal places.
- Answer to each question will be evaluated according to the following making scheme:
 Full Marks : +4 If ONLY the correct numerical value is entered;
 Zero Marks : 0 In all other cases.

- *13. 5.00 mL of 0.10 M oxalic acid solution taken in a conical flask is titrated against NaOH from a burette using phenolphthalein indicator. The volume of NaOH required for the appearance of permanent faint pink color is tabulated below for five experiments. What is the concentration, in molarity, of the NaOH solution?

Exp. No.	Vol. of NaOH (mL)
1	12.5
2	10.5
3	9.0
4	9.0
5	9.0

Sol. 0.11

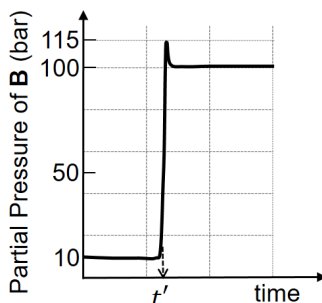
Vol. of NaOH used at equivalent point = 9 mL

So, mequivalent of NaOH used up to end point = $9 \times (M \times 1)$

$$5 \times 0.1 \times 2 = 9 \times M$$

$$M = \frac{1}{9} = 0.11$$

- *14. Consider the reaction $A \rightleftharpoons B$ at 1000 K. At time t' , the temperature of the system was increased to 2000 K and the system was allowed to reach equilibrium. Throughout this experiment the partial pressure of A was maintained at 1 bar. Given below is the plot of the partial pressure of B with time. What is the ratio of standard Gibbs energy of the reaction at 1000 K to that at 2000 K?



Sol. 0.25

$$K_1 \text{ at } 1000 \text{ K} = \frac{10}{1} = 10$$

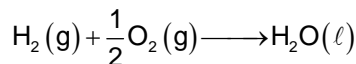
$$K_2 \text{ at } 2000 \text{ K} = \frac{100}{1} = 100$$

$$\Delta G_1^\circ = -RT_1 \ln 10 = -2.303RT_1$$

$$\Delta G_2^\circ = -2.303RT_2 \log 100 = -2 \times 2.303RT_2$$

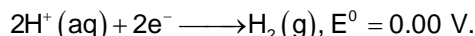
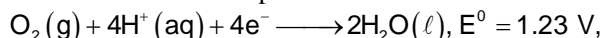
$$\frac{\Delta G_1^\circ}{\Delta G_2^\circ} = \frac{1 \times 1000}{2 \times 2000} = \frac{1}{4} = 0.25$$

15. Consider a 70% efficient hydrogen-oxygen fuel cell working under standard conditions at 1 bar and 298 K. Its cell reaction is



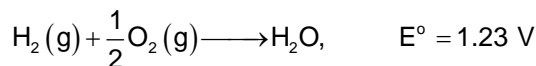
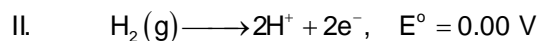
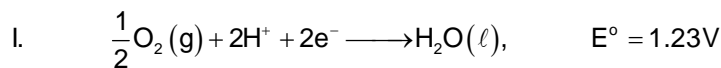
The work derived from the cell on the consumption of 1.0×10^{-3} mol of $\text{H}_2(\text{g})$ is used to compress 1.00 mol of a monoatomic ideal gas in a thermally insulated container. What is the change in the temperature (in K) of the ideal gas?

The standard reduction potentials for the two half-cells are given below.



Use $F = 96500 \text{ C mol}^{-1}$, $R = 8.314 \text{ J mol}^{-1}\text{K}^{-1}$.

Sol. 13.32



$$\Delta G^\circ = -2 \times 96500 \times 1.23 \times 1 \times 10^{-3} \times 0.7 = -166.173 \text{ J}$$

$$W = 166.173 \text{ J}$$

$$W_{\text{adiabatic}} = \frac{nR}{(\gamma - 1)}(T_2 - T_1)$$

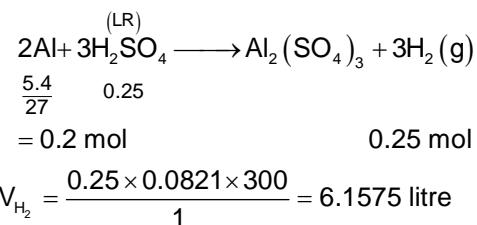
$$166.173 = \frac{1 \times 8.314}{\left(\frac{5}{3} - 1\right)} (T_2 - T_1)$$

$$\Delta T = \frac{166.173}{8.314} \times \frac{2}{3} = 13.32$$

- *16. Aluminium reacts with sulfuric acid to form aluminium sulfate and hydrogen. What is the volume of hydrogen gas in litre (L) produced at 300 K and 1.0 atm pressure, when 5.4 g of aluminium and 50.0 mL of 5.0 M sulfuric acid are combined for the reaction?

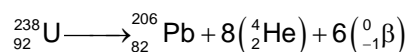
(Use molar mass of aluminium as 27.0 g mol^{-1} , $R = 0.082 \text{ atm L mol}^{-1}\text{K}^{-1}$)

Sol. 06.15



17. ${}^{238}_{92}\text{U}$ is known to undergo radioactive decay to form ${}^{206}_{82}\text{Pb}$ by emitting alpha and beta particles. A rock initially contained $68 \times 10^{-6} \text{ g}$ of ${}^{238}_{92}\text{U}$. If the number of alpha particles that it would emit during its radioactive decay of ${}^{238}_{92}\text{U}$ to ${}^{206}_{82}\text{Pb}$ in three half-lives is $Z \times 10^{18}$, then what is the value of Z?

Sol. 1.20



$$\text{Total moles of uranium} = \frac{68 \times 10^{-6}}{238}$$

$$= 0.286 \times 10^{-6} \text{ mol}$$

So, after 3 half-lives fraction left = 0.125

Fraction of uranium reacted = 0.875

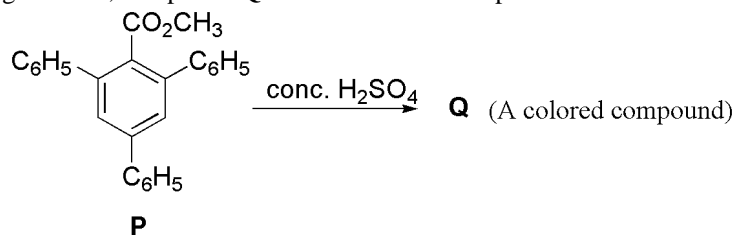
$$\text{So, number of moles of } \alpha \text{ - particles} = \frac{8 \times 68 \times 10^{-6}}{238} \times 0.875 \text{ mol}$$

$$\text{So, number of } \alpha \text{ - particle} = \frac{8 \times 68 \times 10^{-6} \times 0.875 \times 6.023 \times 10^{23}}{238}$$

$$= 12.046 \times 10^{17}$$

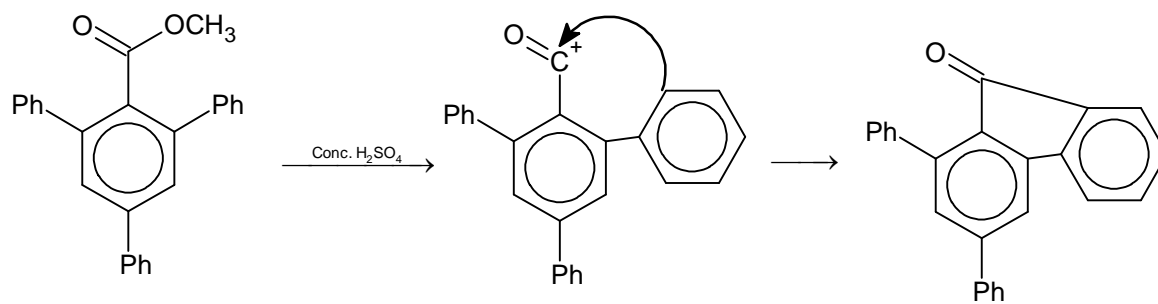
$$= 1.20 \times 10^{18}$$

18. In the following reaction, compound **Q** is obtained from compound **P** via an ionic intermediate.



What is the degree of unsaturation of **Q**?

Sol. 18



Degree of unsaturation = 18

PART III: MATHEMATICS

SECTION 1 (Maximum Marks: 18)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be a evaluated according to the following marking scheme:
Full Marks : +3 if **ONLY** the correct option is chosen;
Zero Marks : 0 if none of these of the options is chosen (i.e. the question is unanswered);
Negative Marks : -1 in all other cases.

- *1. Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c, d denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$. Then the value of $ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$

- (A) 0 (B) 8000 (C) 8080 (D) 16000

Sol. D

$$\begin{aligned} \text{Given } a + b &= -20, ab = -2020, c + d = 20, cd = 2020-09-28 \\ a^2c - ac^2 + a^2d - ad^2 + b^2c - b^2d - bc^2 + b^2d - bd^2 \\ &= (a^2 + b^2)(c + d) - (a + b)(c^2 + d^2) \\ &= ((a + b)^2 - 2ab)(c + d) - (a + b)((c + d)^2 - 2cd) \\ &= (400 - 2 \times 2020)(20) - (-20)(400 - 2 \times 2020) \\ &= 20 \times 800 = 16000 \end{aligned}$$

2. If the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $f(x) = |x|(x - \sin x)$, then which of the following statements is **TRUE?**

- (A) f is one-one, but **NOT** onto (B) f is onto, but **NOT** one-one
 (C) f is **BOTH** one-one and onto (D) f is **NEITHER** one-one **NOR** onto

Sol. C

$f(x)$ is odd, continuous function

$$f(x) = \begin{cases} x(x - \sin x) & , x \geq 0 \\ -x(x - \sin x) & , x < 0 \end{cases}$$

$$\text{for } x \geq 0, f'(x) = 2x - \sin x - x \cos x = x(1 - \cos x) + (x - \sin x) \geq 0$$

$$\text{for } x < 0, f'(x) = -2x + \sin x + x \cos x = x(\cos x - 1) - (x - \sin x) > 0 \text{ as } x < 0$$

\Rightarrow So $f(x)$ strictly increases in $(-\infty, \infty) \Rightarrow f(x)$ is one-one

$$x \rightarrow \infty \quad f(x) \rightarrow \infty$$

$$x \rightarrow -\infty \quad f(x) \rightarrow -\infty. \text{ So } f(x) \text{ is onto}$$

3. Let the function $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = e^{x-1} - e^{-|x-1|} \text{ and } g(x) = \frac{1}{2}(e^{x-1} + e^{1-x}).$$

Then the area of the region in the first quadrant bounded by the curves $y = f(x)$, $y = g(x)$ and $x = 0$ is

- (A) $(2 - \sqrt{3}) + \frac{1}{2}(e - e^{-1})$ (B) $(2 + \sqrt{3}) + \frac{1}{2}(e - e^{-1})$
 (C) $(2 - \sqrt{3}) + \frac{1}{2}(e + e^{-1})$ (D) $(2 + \sqrt{3}) + \frac{1}{2}(e + e^{-1})$

Sol. A

$$f(x) = \begin{cases} 0 & , x \leq 1 \\ e^{x-1} - e^{1-x} & , x > 1 \end{cases}$$

$$g(x) = \frac{1}{2}(e^{x-1} + e^{1-x})$$

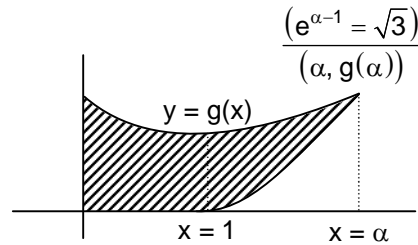
$$f(x) = g(x)$$

$$e^{x-1} - e^{1-x} = \frac{1}{2}(e^{x-1} + e^{1-x})$$

$$e^{x-1} = 3e^{1-x}$$

$$e^{x-1} = \sqrt{3}$$

$$e^{\alpha-1} = \sqrt{3} \text{ . Let } x = \alpha \text{ be P.O.I}$$



$$\text{Required area} = \int_0^1 \frac{1}{2}(e^{x-1} + e^{1-x}) dx + \int_1^\alpha \left(\frac{1}{2}(e^{x-1} + e^{1-x}) - (e^{x-1} - e^{1-x}) \right) dx$$

$$= \frac{1}{2} \left| e^{x-1} - e^{1-x} \right|_0^1 + \left| -\frac{1}{2} e^{x-1} - \frac{3}{2} e^{1-x} \right|_1^\alpha$$

$$= \frac{1}{2} \{ (e^0 - e^0) - (e^{-1} - e^1) \} - \frac{1}{2} \{ e^{\alpha-1} + 3e^{1-\alpha} - (e^0 + 3e^0) \}$$

$$= \frac{1}{2} \left(e - \frac{1}{e} \right) - \frac{1}{2} \left(\sqrt{3} + \frac{3}{\sqrt{3}} - 4 \right)$$

$$= -(\sqrt{3} - 2) + \frac{1}{2} \left(e - \frac{1}{e} \right)$$

$$= (2 - \sqrt{3}) + \frac{1}{2} \left(e - \frac{1}{e} \right)$$

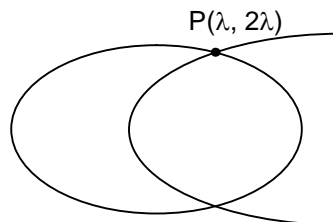
- *4. Let a, b and λ be positive real numbers. Suppose P is an end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and suppose the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the point P . If the tangents to the parabola and the ellipse at the point P are perpendicular to each other, then the eccentricity of the ellipse is

(A) $\frac{1}{\sqrt{2}}$

(B) $\frac{1}{2}$

(C) $\frac{1}{3}$

(D) $\frac{2}{5}$

Sol. AGiven $a, b, \lambda \in \mathbb{R}^+$ As P lies on ellipse $\frac{\lambda^2}{a^2} + \frac{y\lambda^2}{b^2} = 1$ 

So equation of tangent at P to ellipse is, $m = \frac{-\lambda}{a^2} \times \frac{b^2}{2\lambda}$

(Slope of tangent for parabola at P) (Slope of tangent at P for ellipse) = -1

$$\Rightarrow -\frac{b^2}{2a^2} = -1, b^2 = 2a^2$$

$$e = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$$

5. Let C_1 and C_2 be two biased coins such that the probabilities of getting head in a single toss are $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Suppose α is the number of heads that appear when C_1 is tossed twice, independently, and suppose β is the number of heads that appear when C_2 is tossed twice, independently. Then the probability that the roots of the quadratic polynomial $x^2 - \alpha x + \beta$ are real and equal, is

- (A) $\frac{40}{81}$ (B) $\frac{20}{81}$ (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

Sol. B

For C_1 $P(h) = \frac{2}{3}$, for C_2 $P(h) = \frac{1}{3}$, given roots of the equation $x^2 - \alpha x + \beta = 0$ equal real roots

$$\Rightarrow \alpha^2 = 4\beta$$

$$\alpha = 2 \text{ and } \beta = 1 \quad \alpha = \beta = 0$$

$$\text{So required probability} = \left(\frac{2}{3}\right)^2 \cdot {}^2C_1 \cdot \frac{1}{3} \cdot \frac{2}{3} + \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 = \frac{16}{81} + \frac{4}{81} = \frac{20}{81}$$

6. Consider all rectangles lying in the region

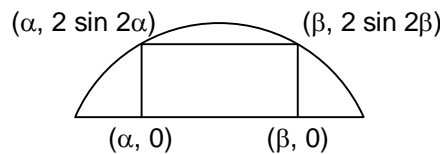
$$\left\{ (x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq \frac{\pi}{2} \text{ and } 0 \leq y \leq 2 \sin(2x) \right\}$$

and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is

- (A) $\frac{3\pi}{2}$ (B) π (C) $\frac{\pi}{2\sqrt{3}}$ (D) $\frac{\pi\sqrt{3}}{2}$

Sol. C

$$0 \leq y \leq 2 \sin 2x, 0 < x < \frac{\pi}{2}$$



for rectangle $2 \sin 2\alpha = 2 \sin 2\beta \Rightarrow \alpha + \beta = \frac{\pi}{2}$

$$\text{Perimeter} = 2(\beta - \alpha) + 4 \sin 2\beta = 2\left(\beta - \left(\frac{\pi}{2} - \beta\right)\right) + 4 \sin 2\beta$$

$$= 4\beta - \pi + 4 \sin 2\beta$$

$$\frac{dp}{d\beta} = 4 + 8\cos 2\beta = 0, \cos 2\beta = -\frac{1}{2}, \beta = \frac{\pi}{3}$$

$$\frac{d^2p}{d\beta^2} = -16\sin 2\beta < 0 \text{ so P is max}$$

$$\begin{aligned} \text{So, area of rectangle} &= (\beta - \alpha)2 \sin 2\beta = \left(\frac{\pi}{3} - \frac{\pi}{6}\right) \cdot 2 \sin\left(\frac{2\pi}{3}\right) \\ &= \frac{\pi}{6} \times 2 \cdot \frac{\sqrt{3}}{2} = \frac{\pi}{2\sqrt{3}} \end{aligned}$$

SECTION 2 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) the correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

<i>Full Marks</i>	: +4 if only (all) the correct option(s) is(are) chosen;
<i>Partial Marks</i>	: +3 if all the four options are correct but ONLY three options are chosen;
<i>Partial Marks</i>	: +2 If three or more options are correct but ONLY two options are chosen, both of which are correct;
<i>Partial Marks</i>	: + 1 If two or more options are correct but ONLY one option is chosen and it is a correct option;
<i>Zero Marks</i>	: 0 if none of the options is chosen (i.e. the question is unanswered);
<i>Negative Marks</i>	: -2 in all other cases.

7. Let the function $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^3 - x^2 + (x - 1) \sin x$ and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary function. Let $fg : \mathbb{R} \rightarrow \mathbb{R}$ be the product function defined by $(fg)(x) = f(x)g(x)$. Then which of the following statements is/are TRUE?
- (A) If g is continuous at $x = 1$, then fg is differentiable at $x = 1$
- (B) If fg is differentiable at $x = 1$, then g is continuous at $x = 1$
- (C) If g is differentiable at $x = 1$, then fg is differentiable at $x = 1$
- (D) If fg is differentiable at $x = 1$, then g is differentiable at $x = 1$

Sol. A, C

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = x^3 - x^2 + (x - 1) \sin x$$

$$g : \mathbb{R} \rightarrow \mathbb{R}$$

- (A) If g is continuous at $x = 1$ then fg is differentiable

$$\text{Let } h(x) = f(x) \cdot g(x)$$

$$\text{RHD} = h'(1^+) = \lim_{h \rightarrow 0} \frac{f(1+h)g(1+h) - f(1)g(1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(1+h)g(1+h) - 0}{h}$$

$$= \lim_{h \rightarrow 0} g(1+h) \cdot \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$= g(1) f'(1)$$

LHD at $x = 1$

$$= \lim_{h \rightarrow 0} \frac{f(1-h)g(1-h) - f(1)g(1)}{-h}$$

$$= \lim_{h \rightarrow 0} g(1-h) \cdot \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h} = g(1) f'(1)$$

So, $h(x)$ is differentiable at $x = 1$

(B) Given $h(x) = f(x) \cdot g(x)$ is differentiable

$$\lim_{h \rightarrow 0} \frac{f(1+h)g(1+h) - f(1)g(1)}{h} = \lim_{h \rightarrow 0} \frac{f(1-h)g(1-h) - f(1)g(1)}{-h}$$

$$\lim_{h \rightarrow 0} g(1+h) \cdot f'(1) = \lim_{h \rightarrow 0} g(1-h) \cdot f'(1)$$

$f'(1) \neq 0$ and $g(1)$ is not define

So, can not comment over continuity and differentiability

(C) Given $g(x)$ is differentiable

So, $h(x) = f(x) g(x)$

$h'(x) = f'(x) g(x) + g'(x) f(x)$, as $g(x)$ is differentiable

$= f'(1) g(1) + 0$ will exist

(D) Same as for B

$$\lim_{h \rightarrow 0} g(1+h) = \lim_{h \rightarrow 0} g(1-h)$$

can not say about differentiability of the $g(x)$

8. Let M be a 3×3 invertible matrix with real entries and let I denote the 3×3 identity matrix. If

$M^{-1} = \text{adj}(\text{adj } M)$, then which of the following statements is/are ALWAYS TRUE?

(A) $M = I$

(B) $\det M = 1$

(C) $M^2 = I$

(D) $(\text{adj } M)^2 = I$

Sol. B, C, D

$$M^{-1} = \text{adj}(\text{adj } M)$$

$$M \text{ adj } M = |M| I$$

$$= |M| M$$

replace M by $\text{adj } M$

$$\Rightarrow |M^{-1}| = ||M| M|$$

$$(\text{adj } M) \text{ adj}(\text{adj } M) = |\text{adj } M| I = |M|^2 I$$

$$\frac{1}{|M|} = |M|^3 |M| \Rightarrow |M| = 1 \dots (1)$$

Multiplying by M

$$\frac{\text{adj } M}{|M|} = |M| M$$

$$M(\text{adj } M) \text{ adj}(\text{adj } M) = |M|^2 M$$

$$\text{adj } M = |M|^2 M \Rightarrow \text{adj } M = M \text{ as } |M| = 1$$

$$|M| \text{ adj}(\text{adj } M) = |M|^2 M$$

$$\Rightarrow M (\text{adj } M) = M^2$$

$$\text{adj}(\text{adj } M) = |M| M$$

$$\Rightarrow M^2 = I$$

$$\text{adj } M = M \text{ as } |M| = 1$$

$$(\text{adj } M)^2 = I \text{ (as } \text{adj } M = M)$$

*9. Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are TRUE?

(A) $\left| z + \frac{1}{2} \right| \leq \frac{1}{2}$ for all $z \in S$

(B) $|z| \leq 2$ for all $z \in S$

(C) $\left| z + \frac{1}{2} \right| \geq \frac{1}{2}$ for all $z \in S$

(D) The set S has exactly four elements

Sol. B, C

$$|z^2 + z + 1| = 1$$

$$\left| \left(z + \frac{1}{2} \right)^2 + \frac{3}{4} \right| = 1$$

Using Δ inequality

$$\left| z + \frac{1}{2} \right|^2 + \frac{3}{4} \geq |z^2 + z + 1| = 1$$

$$\left| z + \frac{1}{2} \right|^2 \geq \frac{1}{4}$$

$$\left| z + \frac{1}{2} \right| \geq \frac{1}{2} \quad \dots (1)$$

$$||z^2 + z| - 1| \leq 1$$

$$0 \leq |z^2 + z| \leq 2 \Rightarrow |z^2 + z| \leq 2$$

$$|z|^2 - |z| \leq 2 \Rightarrow |z|^2 - |z| - 2 \leq 0$$

$$(|z| - 2)(|z| + 1) \leq 0$$

$$|z| \leq 2$$

- *10. Let x , y and z be positive real numbers. Suppose x , y and z are the lengths of the sides of a triangle opposite to its angles X , Y and Z , respectively. If

$$\tan \frac{X}{2} + \tan \frac{Z}{2} = \frac{2y}{x + y + z},$$

then which of the following statements is/are TRUE?

(A) $2Y = X + Z$

(B) $Y = X + Z$

(C) $\tan \frac{X}{2} = \frac{x}{y + z}$

(D) $x^2 + z^2 - y^2 = xz$

Sol. **B, C**

$$\tan \frac{X}{2} + \tan \frac{Z}{2} = \frac{2y}{2x + y + z}$$

$$\Rightarrow \frac{\Delta}{s(s-x)} + \frac{\Delta}{s(s-z)} = \frac{2y}{2s} \Rightarrow \Delta = (s-x)(s-z)$$

$$\Rightarrow s(s-y) = (s-x)(s-z) \Rightarrow z^2 + x^2 = y^2$$

$$\Rightarrow \text{Triangle is right angled, } y = \frac{\pi}{2}$$

$$\Rightarrow Y = X + Z$$

$$\tan \frac{X}{2} = \frac{\Delta}{s(s-x)} = \frac{\frac{1}{2}xz}{\left(\frac{x+y+z}{2}\right)\left(\frac{y+z-x}{2}\right)} = \frac{x}{y+z}$$

11. Let L_1 and L_2 be the following straight lines.

$$L_1 : \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3} \quad \text{and} \quad L_2 : \frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1}$$

Suppose the straight line

$$L : \frac{x-\alpha}{l} = \frac{y-1}{m} = \frac{z-\gamma}{-2}$$

lies in the plane containing L_1 and L_2 , and passes through the point of intersection of L_1 and L_2 . If the line L bisects the acute angle between the lines L_1 and L_2 , then which of the following statements is/are TRUE?

(A) $\alpha - \gamma = 3$

(B) $l + m = 2$

(C) $\alpha - \gamma = 1$

(D) $l + m = 0$

Sol. A, B

$$\text{DCs of } L_1 : \frac{1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{3}{\sqrt{11}}$$

$$\text{DCs of } L_2 : \frac{-3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{1}{\sqrt{11}}$$

$$\text{Since } \frac{1}{\sqrt{11}} \left(\frac{-3}{\sqrt{11}} \right) + \frac{-1}{\sqrt{11}} \left(\frac{-1}{\sqrt{11}} \right) + \frac{3}{\sqrt{11}} \frac{1}{\sqrt{11}} = \frac{1}{11} > 0$$

$$\text{So DCs of acute angle bisectors are } \frac{-2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{4}{\sqrt{11}}$$

D.r.s can be written as $1, 1, -2 \Rightarrow \ell = m = 1$

L_1 and L_2 intersect at $(1, 0, 1)$ which also lies on bisector L. Point on L having y-coordinate equal to 1 is

$$(2, 1, -1) \Rightarrow \alpha = 2, \gamma = -1$$

12. Which of the following inequalities is/are TRUE?

$$(A) \int_0^1 x \cos x dx \geq \frac{3}{8} \quad (B) \int_0^1 x \sin x dx \geq \frac{3}{10} \quad (C) \int_0^1 x^2 \cos x dx \geq \frac{1}{2} \quad (D) \int_0^1 x^2 \sin x dx \geq \frac{2}{9}$$

Sol. A, B, D

$$(A) \int_0^1 x \cos x dx \geq \int_0^1 x \left(1 - \frac{x^2}{2!} \right) dx$$

$$\geq \frac{3}{8}$$

$$(B) \int_0^1 x \sin x dx \geq \int_0^1 x \left(x - \frac{x^3}{3!} \right) dx$$

$$\geq \frac{3}{10}$$

$$(C) \int_0^1 x^2 \cos x dx \leq \int_0^1 x^2 dx$$

$$\leq \frac{1}{3}$$

$$(D) \int_0^1 x^2 \sin x dx \geq \int_0^1 x^2 \left(x - \frac{x^3}{3!} \right) dx$$

$$\geq \frac{2}{9}$$

SECTION 3 (Maximum Marks: 24)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical values of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value of **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:
Full Marks : +4 If **ONLY** the correct numerical value is entered;
Zero Marks : 0 in all other cases.

- *13. Let m be the minimum possible value of $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is _____

Sol. **8**

$$\frac{3^{y_1} + 3^{y_2} + 3^{y_3}}{3} \geq 3^{\frac{y_1+y_2+y_3}{3}} \quad (\text{Using AM} \geq \text{GM})$$

$$\Rightarrow \log_3(3^{y_1} + 3^{y_2} + 3^{y_3}) \geq \log_3 3^4$$

$$\geq 4$$

$$\Rightarrow m = 4$$

$$\text{Similarly, } \frac{x_1 + x_2 + x_3}{3} \geq (x_1 x_2 x_3)^{\frac{1}{3}}$$

$$\log_3 3 \geq \frac{1}{3}(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$$

$$3 \geq \log_3 x_1 + \log_3 x_2 + \log_3 x_3$$

$$\Rightarrow M = 3$$

$$\log_2 m^3 + \log_3 M^2 = 8$$

- *14. Let a_1, a_2, a_3, \dots be a sequence of positive integers in arithmetic progression with common difference 2. Also, let b_1, b_2, b_3, \dots be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c , for which the equality

$$2(a_1 + a_2 + \dots + a_n) = b_1 + b_2 + \dots + b_n$$

holds for some positive integer n , is _____

Sol. **1**

$$2(a_1 + a_2 + \dots + a_n) = b_1 + b_2 + \dots + b_n$$

$$\Rightarrow 2\left(\frac{n}{2}(2c + (n-1)2)\right) = c\left(\frac{2^n - 1}{2 - 1}\right)$$

$$c = \frac{2n(n-1)}{2^n - 1 - 2n}$$

$$2^{x-2} > x^2 - x \quad \text{if } x \geq 8$$

$$2^x - 1 > 2x \quad \text{if } x > 3$$

$$\text{So, } c < \frac{2^{n-1}}{2^n - 1} \quad \forall n \geq 8$$

as $c \geq 1$ values of c are not possible for $n \geq 8$

possible values of $n = 3, 4, 5, 6, 7$; only satisfy when $n = 3$ and $c = 12$

*15. Let $f : [0, 2] \rightarrow \mathbb{R}$ be the function defined by

$$f(x) = (3 - \sin(2\pi x)) \sin\left(\pi x - \frac{\pi}{4}\right) - \sin\left(3\pi x + \frac{\pi}{4}\right).$$

If $\alpha, \beta \in [0, 2]$ are such that $\{x \in [0, 2] : f(x) \geq 0\} = [\alpha, \beta]$, then the value of $\beta - \alpha$ is _____

Sol. 1

$$\begin{aligned} & (3 - \sin 2\pi x) \sin\left(\pi x - \frac{\pi}{4}\right) - \sin\left(3\pi x + \frac{\pi}{4}\right) \\ &= (3 - \sin 2\pi x) \sin\left(\pi x - \frac{\pi}{4}\right) + \sin\left(3\pi x - \frac{3\pi}{4}\right) \\ &= \left(3 - \sin 2\pi x + 3 - 4 \sin^2\left(\pi x - \frac{\pi}{4}\right)\right) \sin\left(\pi x - \frac{\pi}{4}\right) \\ &= \left(3 - \sin 2\pi x + 3 - 4 \sin^2\left(\pi x - \frac{\pi}{4}\right)\right) \sin\left(\pi x - \frac{\pi}{4}\right) \\ &\Rightarrow \sin\left(\pi x - \frac{\pi}{4}\right) \geq 0 \\ &\Rightarrow \pi \geq \pi x - \frac{\pi}{4} \geq 0 \quad (\because x \in [0, 2]) \\ &\Rightarrow \frac{5}{4} \geq x \geq \frac{1}{4} \Rightarrow \beta - \alpha = 1 \end{aligned}$$

16. In a triangle PQR, let $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$. If

$$|\vec{a}| = 3, |\vec{b}| = 4 \text{ and } \frac{\vec{a} \cdot (\vec{c} - \vec{b})}{\vec{c} \cdot (\vec{a} - \vec{b})} = \frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|},$$

then the value of $|\vec{a} \times \vec{b}|^2$ is _____

Sol. 108

$$\text{Equation reduces to } (4\vec{a} + 3\vec{b}) \cdot \vec{c} = 7\vec{a} \cdot \vec{b}$$

$$\text{But } \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$$\text{So, } (4\vec{a} + 3\vec{b}) \cdot (\vec{a} + \vec{b}) = -7\vec{a} \cdot \vec{b}$$

$$\vec{a} \cdot \vec{b} = -6$$

$$|\vec{a} \times \vec{b}|^2 = 108$$

17. For a polynomial $g(x)$ with real coefficients, let m_g denote the number of distinct real roots of $g(x)$. Suppose S in the set of polynomials with real coefficients defined by

$$S = \{(x^2 - 1)^2(a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

For a polynomial f , let f' and f'' denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$, where $f \in S$, is _____

Sol. 5

$$f = (x^2 - 1)^2(a_0 + a_1x + a_2x^2 + a_3x^3)$$

$$= (x + 1)^2(x - 1)^2(a_0 + a_1x + a_2x^2 + a_3x^3)$$

For min value of $m_f + m_{f'}$, $a_0 + a_1x + a_2x^2 + a_3x^3$ must have only 1 real root which is 1 or -1, then $m_f = 3$ & $m_{f'} = 2 \Rightarrow$ Answer = 5

18. Let e denote the base of the natural logarithm. The value of the real number a for which the right hand limit

$$\lim_{x \rightarrow 0^+} \frac{(1-x)^{1/x} - e^{-1}}{x^a}$$

is equal to a nonzero real number, is _____

Sol. 1

$$\lim_{x \rightarrow 0^+} \frac{(1-x)^{1/x} - 1}{x^a} = \lim_{x \rightarrow 0^+} \frac{e^{\frac{\ln(1-x)}{x}} - 1}{x^a}$$

$$= \lim_{x \rightarrow 0^+} \frac{1}{e} \frac{\left(e^{\frac{-x - \frac{x^2}{2} + 1}{x}} - 1 \right)}{x^a} = \lim_{x \rightarrow 0} \frac{1}{e} \left(\frac{e^{\frac{x}{2} - 1}}{x^a} \right)$$

For value of limit to be a non-zero real number $a = 1$

END OF THE QUESTION PAPER